COMBINATORIAL OPTIMIZATION
Written test of January 26, 2016

1. Distinguish between true and false:

Group A

a) A complete bipartite graph is always a regular graph.

b) A totally unimodular matrix cannot have elements that differ from 0 or 1.

c) If a regular graph is Hamiltonian, then the degree of its nodes is necessarily even.

d) The largest stable set of a bipartite graph has always as many nodes as the least number of arcs contained in an edge-cover if the graph.

true	*false*
true	*false*
true	*false*
true	*false*

Group B

a) The largest stable set in a bipartite graph with bipartition V_1, V_2 has always $\max\{|V_1|, |V_2|\}$ nodes.

b) If an *mxn* matrix is totally unimodular, then also the matrix obtained by adding to it an identity *mxm* matrix is totally unimodular.

c) Every Hamiltonian graph with an even number of nodes has a perfect matching.

d) The graph obtained from any symmetric graph *G* by splitting every arc with a node is surely bipartite.

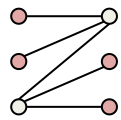
true	*false*
true	*false*
true	*false*
true	*false*

Group A

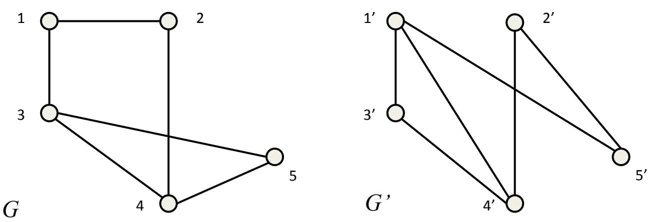
The first three sentences are false. Sentence (*a*) for example is false for $K_{2,3}$. Sentence (*b*) is false because the node-arc incidence matrix of a directed graph is totally unimodular and has various negative elements. A counterexample to sentence (*c*) is a cube. Sentence (*d*) is true: in fact, it is König's Theorem.

Group B

Sentence (*a*) is false: a counterexample is the small graph drawn aside. Sentence (*b*) is true for a well known theorem. Sentence (*c*) is true as well: the perfect matching is obtained by taking the even (or the odd) arcs encountered in the Hamiltonian circuit. Also sentence (*d*) is true: in fact, after interposing a node in each arc, every odd cycle (if any) of *G* becomes even.



2. Decide whether the graphs *G*, *G'* in the figure are or not isomorphic.



3. Describe a system of linear inequalities among 20 variables, the solutions of which in $\{0, 1\}^{20}$ represent all possible isomorphisms between the given graphs *G*, *G'*.

Group A

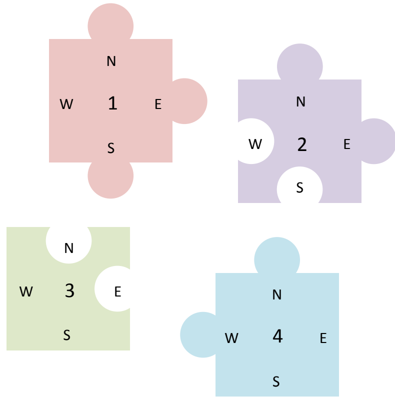
The inequalities are defined for the variables $x_{11'}, x_{12'}, \ldots, x_{55'}$. Generic variable $x_{ij'}$ will take value 1 if the isomorphism associates node *i* of graph $G = (V, E)$ with node *j'* of $G' = (V', E')$. The inequalities have the general form $x_{ij} + x_{hk} \leq 1$ if $ih \in E$ e $j'k' \notin E'$, or if $ih \notin E$ e $j'k' \in E'$. For instance

$x_{11'} + x_{32'} \leq 1$ because $13 \in E$ e $1'2' \notin E'$

$x_{11'} + x_{44'} \leq 1$ because $14 \notin E$ e $1'4' \in E'$

etcetera.

4. We are given a few pieces of a jigsaw puzzle. Every piece can share from two to four sides with other pieces: for example, piece 1 in the figure on the right can share three sides with other pieces, piece 2 four sides and so on. Let us assume that pieces are already oriented, meaning that for each piece we know which are the north, east, south and west side. Let then n_{ij}, e_{ij}, s_{ij}, w_{ij} be 0-1 variables associated with the north, east, south and west sides of piece i, with the following meaning: if, say, $n_{ij} = 1$, then the north side of piece i is connected to the south side of piece j; if $n_{ij} = 0$, it is not. Clearly, variable n_{ij} (or e_{ij} and the like) is only defined for pairs (i, j) of compatible pieces. For instance, referring to the figure, variable e_{12} is defined, because the east side of piece 1 fits with the west side of piece 2; on the contrary, variable s_{12} is not defined because the south side of piece 1 does not fit with the north side of piece 2. Note that $e_{12} = w_{21}$ (in fact the east and west sides of the two pieces match in both directions), so to avoid useless duplications we can limit variable definition to $i < j$.



Write a set of inequalities that must be fulfilled by the variables so defined when these encode a way to fit pieces as required by jigsaw puzzles; define also an objective function that counts the total number of pairs matched by such a solution.

First of all, a side of the generic piece i can be associated to no more than one side of another piece: hence

$$\sum_{j>i} n_{ij} \leq 1; \qquad \sum_{j>i} e_{ij} \leq 1; \qquad \sum_{j>i} s_{ij} \leq 1; \qquad \sum_{j>i} w_{ij} \leq 1;$$

where we write the inequalities only for the sides of piece i that can actually be matched (for example, for piece 1 we will not write the last inequality).

Moreover, if two sides of two pieces are matched, then each of those pieces cannot have another side matched with a side of the other piece:

$$n_{ij} + e_{ij} + s_{ij} + w_{ij} \leq 1 \qquad \text{for any pair of pieces } i, j \text{ with } i < j$$

The total number of side pairs matched equals the sum of all the variables so defined.

5. Consider the problem of finding the largest stable set in the graph G of Problem 2, and assume you want to solve this problem by branch-and-bound. Suppose that you begun with determining a first feasible solution by the greedy algorithm, and that you then started a dichotomy on variable x_1 which, set to 1, indicates that node 1 is in the stable set, and set to 0 indicates instead that node 1 does not belong to that set.

Compute an upper bound in both branches by solving (by enumeration) an edge-covering problem (why is this an upper bound?). In this way, decide whether both branches can be closed or it is necessary to open a further branch.

Group A

As all nodes have unitary weights, the greedy method proceeds by arbitrarily selecting the current node. A possible solution is therefore $S = \{1, 4\}$, of value 2.

Setting $x_1 = 1$ implies setting $x_2 = x_3 = 0$. The largest stable set in this branch is therefore obtained by adding to node 1 the largest stable set in the subgraph G_1 induced by $\{4, 5\}$. But the least edge-cover of G_1 contains one arc only, so the largest stable set in G_1 has ≤ 1 nodes (this is in fact an upper bound by weak duality). This means that the largest stable set of G, given $x_1 = 1$, contains $\leq 1 + 1 = 2$ nodes. We then conclude that on this side of our dichotomy we cannot get any improvement of the initial solution.

On the other hand, setting $x_1 = 0$ implies the elimination of node 1 from graph G: let then G_0 be the subgraph induced by $\{2, 3, 4, 5\}$. As before, the elements of a largest stable set of G_0 are no more than those in a minimum edge-cover of G_0. This edge-cover contains 2 arcs (24 and 35), thus the largest stable set in G_0 has ≤ 2 nodes. Consequently the largest stable set of G, given $x_1 = 0$, has $\leq 0 + 2 = 2$ nodes. Also this side of the dichotomy cannot improve the initial solution, which turns then out to be optimal.